# VC Dimension, VC Density, and an Application to Algebraically Closed Valued Fields

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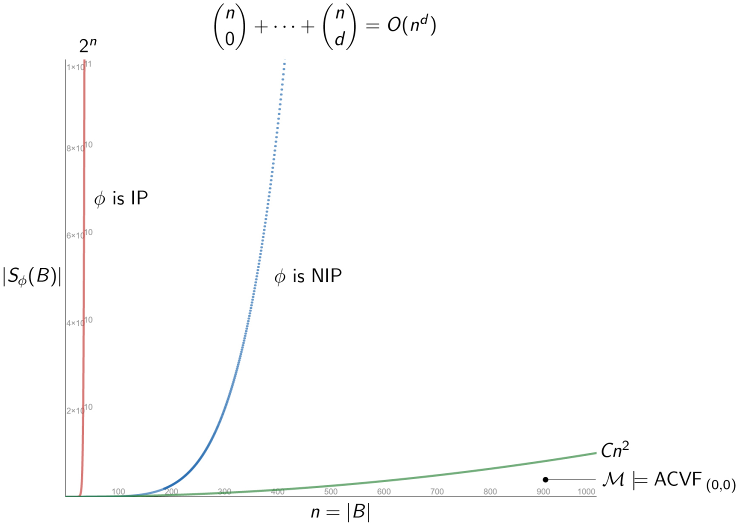
# Counting Types

Let $\mathcal{L}$ be a language, $\mathcal{M}$ an $\mathcal{L}$-structure, $\phi(x, y) \in \mathcal{L}$ with $|x| = 1$, and $B \subseteq M^{|y|}$.



# References

A. Aschenbrenner, A. Dolich, D. Haskell, H. D. Macpherson, and S. Starchenko, *Vapnik-Chervonenkis density in some theories without the independence property, I*, Trans. Amer. Math. Soc. 368 (2016), 5889-5949.

V. Guingona, *On VC-density in VC-minimal theories*, arXiv:1409.8060 [math.LO].

P. Simon, *A Guide to NIP Theories*, Cambridge University Press (2015).

# Set Systems

### Definition

Let $X$ be a set and $\mathcal{S} \subseteq \mathcal{P}(X)$. We call the pair $(X, \mathcal{S})$ a *set system*.

### Definition

Given $A \subseteq X$, define

$$\mathcal{S} \cap A = \{B \cap A : B \in \mathcal{S}\}.$$

We say $A$ is *shattered* by $\mathcal{S}$ iff: $\mathcal{S} \cap A = \mathcal{P}(A)$.

# The Shatter Function and VC Dimension

**Definition**

The function $\pi_{\mathcal{S}} : \omega \to \omega$ given by

$$\pi_{\mathcal{S}}(n) = \max\{|\mathcal{S} \cap A| : A \in [X]^n\}$$

is called the *shatter function* of $\mathcal{S}$.

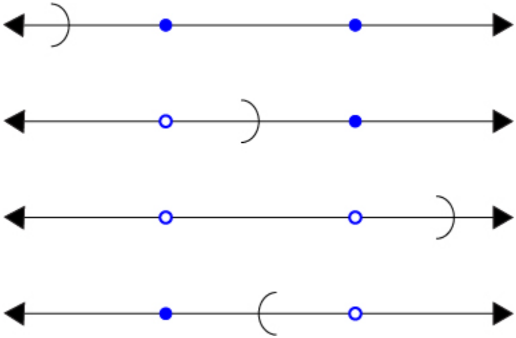
**Definition**

The *Vapnik-Chervonenkis (VC) dimension* of $\mathcal{S}$ is

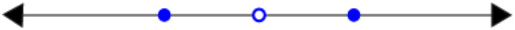
$$\begin{aligned} \mathrm{VC}(\mathcal{S}) &= \sup\{n < \omega : \mathcal{S} \text{ shatters some } A \in [X]^n\} \\ &= \sup\{n < \omega : \pi_{\mathcal{S}}(n) = 2^n\}. \end{aligned}$$

# Example: $X = \mathbb{R}$, $\mathcal{S} =$ Half-Spaces

VC($\mathcal{S}$) $\geq$ 2:

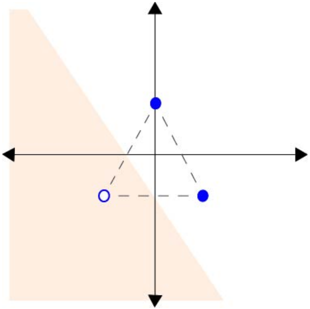


VC($\mathcal{S}$) $<$ 3:



# Example: $X = \mathbb{R}^2$, $\mathcal{S} =$ Half-Spaces

$\mathrm{VC}(\mathcal{S}) \geq 3$ :



$\mathrm{VC}(\mathcal{S}) < 4$ :

# VC Density and the Sauer-Shelah Lemma

## Definition

The *VC density* of $\mathcal{S}$ is

$$\mathrm{vc}(\mathcal{S}) = \inf \left\{ r \in \mathbb{R}^{>0} : \pi_{\mathcal{S}}(n) = O(n^r) \right\} = \limsup_{n \to \omega} \frac{\log \pi(n)}{\log n}.$$

## Lemma (Sauer-Shelah)

*If* $\mathrm{VC}(\mathcal{S}) = d < \omega$, *then for all* $n \geq d$, *we have*

$$\pi_{\mathcal{S}}(n) \leq \binom{n}{0} + \cdots + \binom{n}{d} = O(n^d).$$

## Corollary

$\mathrm{vc}(\mathcal{S}) \leq \mathrm{VC}(\mathcal{S})$.

# Example: When $\mathcal{S}$ is "uniform," VC dimension and VC density agree.

Let $X$ be an infinite set and $\mathcal{S} = [X]^{\leq d}$ for some $d < \omega$.

We have

$$\pi_{\mathcal{S}}(n) = \binom{n}{0} + \cdots + \binom{n}{d},$$

so

$$\mathrm{VC}(\mathcal{S}) = \mathrm{vc}(\mathcal{S}) = d.$$

# Example: VC dimension is more susceptible to local anomalies than VC density.

Let $X = \omega$, $m < \omega$, and $\mathcal{S} = \mathcal{P}(m)$.

It follows that

$$\pi_{\mathcal{S}}(n) = \begin{cases} 2^n & \text{if } n \leq m \\ 2^m & \text{otherwise.} \end{cases}$$

So

$$\mathrm{VC}(\mathcal{S}) = m$$

and

$$\mathrm{vc}(\mathcal{S}) = \limsup_{n \to \omega} \frac{\log 2^m}{\log n} = 0.$$

# The Dual Shatter Function

## Definition

Given $A_1, ..., A_n \subseteq X$, let $S(A_1, ..., A_n)$ denote the set of nonempty atoms in the Boolean algebra generated by $A_1, ..., A_n$. That is

$$S(A_1, \cdots, A_n) = \left\{ \bigcap_{i=1}^{n} A_i^{\sigma(i)} : \sigma \in {}^{n}2 \right\} \setminus \varnothing$$

where $A_i^1 = A_i$ and $A_i^0 = X \setminus A_i$.

## Definition

The function $\pi^*_{\mathcal{S}} : \omega \to \omega$ given by

$$\pi^*_{\mathcal{S}}(n) = \max\{|S(A_1, ..., A_n)| : A_1, ...A_n \in \mathcal{S}\}$$

is called the *dual shatter function* of $\mathcal{S}$.

# Independence Dimension and Dual VC Density

## Definition

The *independence dimension* (a.k.a. dual VC dimension) of $\mathcal{S}$ is

$$\mathrm{IN}(\mathcal{S}) = \mathrm{VC}^*(\mathcal{S}) = \sup \{ n < \omega : \pi^*_{\mathcal{S}}(n) = 2^n \} .$$

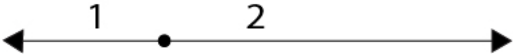
## Definition

The *dual VC density* of $\mathcal{S}$ is

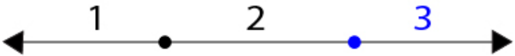
$$\mathrm{vc}^*(\mathcal{S}) = \inf \left\{ r \in \mathbb{R}^{>0} : \pi^*_{\mathcal{S}}(n) = O(n^r) \right\} .$$

# Example: $X = \mathbb{R}$, $\mathcal{S}$ = Half-Spaces

$\mathrm{IN}(\mathcal{S}) \geq 1$:

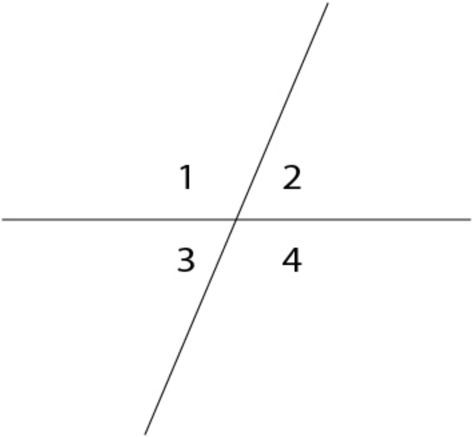



$\mathrm{IN}(\mathcal{S}) < 2$:



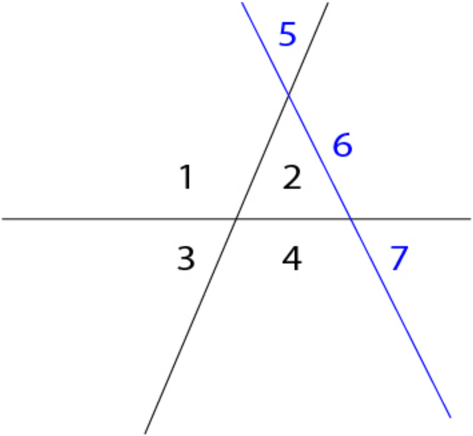
# Example: $X = \mathbb{R}^2$, $\mathcal{S} =$ Half-Spaces

$\mathsf{IN}(\mathcal{S}) \geq 2$ :



# Example: $X = \mathbb{R}^2$, $\mathcal{S} =$ Half-Spaces

$\mathrm{IN}(\mathcal{S}) < 3$ :



# Breadth and Directed Systems

## Definition

Suppose there is a $t < \omega$ such that for all $n > t$, if $\mathcal{A} \in [\mathcal{S}]^n$ and $\bigcap \mathcal{A} \neq \varnothing$, then there is a subfamily $\mathcal{B} \in [\mathcal{A}]^t$ such that $\bigcap \mathcal{A} = \bigcap \mathcal{B}$. We call the least such $t$ the *breadth* of $\mathcal{S}$ and denote it as $\operatorname{breadth}(\mathcal{S})$.

## Definition

We call $\mathcal{S}$ *directed* iff: $\operatorname{breadth}(\mathcal{S}) = 1$.

Example: Let $(K, \Gamma, v)$ be a valued field.
The set system $(X, \mathcal{S})$ where $X = K$ and

$$\mathcal{S} = \{B_\gamma(a) : a \in K, \gamma \in \Gamma\} \cup \{\overline{B}_\gamma(a) : a \in K, \gamma \in \Gamma\}$$

is directed.

# Independence Dimension is Bounded by Breadth

**Lemma**

$\mathrm{IN}(\mathcal{S}) \leq \mathrm{breadth}(\mathcal{S})$.

Proof: Suppose $0 < n = \mathrm{IN}(\mathcal{S}) < \omega$.

There exists $\mathcal{A} \in [\mathcal{S}]^n$ such that $S(\mathcal{A}) = 2^n$.

It follows that $\bigcap \mathcal{A} \neq \varnothing$.

Let $A_0 \in \mathcal{A}$, $\mathcal{B} = \mathcal{A} \setminus A_0$.

Since $(X \setminus A_0) \cap (\bigcap \mathcal{B}) \neq \varnothing$, we have $\bigcap \mathcal{A} \neq \bigcap \mathcal{B}$.

It follows that $\mathrm{breadth}(\mathcal{S}) > n - 1$.

# Set Systems in a Model-Theoretic Context

Consider a sorted language $\mathcal{L}$ with sorts indexed by $I$.

Let $\mathcal{M}$ be an $\mathcal{L}$-structure with domains $(M_i : i \in I)$.

**Definition**

Given an $\mathcal{L}$-formula $\phi(x, y)$ where $x = (x_1^{i_1}, ..., x_s^{i_s})$ and $y = (y_1^{j_1}, ..., y_t^{j_t})$, define

$$\mathcal{S}_\phi = \{\phi(X, b) : b \in Y\}$$

where $X = M_{i_1} \times \cdots \times M_{i_s}$ and $Y = M_{j_1} \times \cdots \times M_{j_t}$.

It follows that $(X, \mathcal{S}_\phi)$ is a set system. To ease notation, we let:

$\pi_\phi$ denote $\pi_{\mathcal{S}_\phi}$, $VC(\phi)$ denote $VC(\mathcal{S}_\phi)$, and $vc(\phi)$ denote $vc(\mathcal{S}_\phi)$.

Similarly, we use $\pi^*_\phi$ for $\pi^*_{\mathcal{S}_\phi}$, $\mathrm{VC}^*(\phi)$ for $\mathrm{VC}^*(\mathcal{S}_\phi)$, and $\mathrm{vc}^*(\phi)$ for $\mathrm{vc}^*(\mathcal{S}_\phi)$.

# The dual shatter function of $\phi$ is really counting $\phi$-types.

By definition, we have $\pi^*_\phi(n) = \max\{|S(\phi(X,b) : b \in B)| : B \in [Y]^n\}$.

Let $B \in [Y]^n$. Recall that

$$S(\phi(X,b) : b \in B) = \left\{ \bigcap_{b \in B} \phi^{\sigma(b)}(X,b) : \sigma \in {}^B 2 \right\} \setminus \varnothing.$$

There is a bijection

$$S(\phi(X,b) : b \in B) \longrightarrow \left\{ \mathrm{tp}_\phi(a/B) : a \in X \right\} = S_\phi(B)$$

given by

$$\bigcap_{b \in B} \phi^{\sigma(b)}(X,b) \longmapsto \left\{ \phi^{\sigma(b)}(x,b) : b \in B \right\}.$$

It follows that

$$|S(\phi(X,b) : b \in B)| = |S_\phi(B)|.$$

# The Dual of a Formula

## Definition

We call a formula $\phi(x; y)$ a *partitioned formula* with *object variable(s)* $x = (x_1, ..., x_s)$ and *parameter variable(s)* $y = (y_1, ..., y_t)$.

## Definition

We let $\phi^*(y; x)$ denote the *dual* of $\phi(x; y)$, meaning $\phi^*(y; x)$ is $\phi(x; y)$ but we view $y$ as the object and $x$ as the parameter.

It follows that

$$\begin{aligned} \mathcal{S}_{\phi^*} &= \{\phi^*(Y, a) : a \in X\} \\ &= \{\phi(a, Y) : a \in X\}. \end{aligned}$$

# The shatter function of $\phi^*$ is also counting $\phi$-types.

By definition, we have $\pi_{\phi^*}(n) = \max\left\{|\mathcal{S}_{\phi^*} \cap B| : B \in [Y]^n\right\}$.

Let $B \in [Y]^n$. It follows that

$$\begin{aligned}\mathcal{S}_{\phi^*} \cap B &= \{\phi^*(B, a) : a \in X\} \\ &= \{\phi(a, B) : a \in X\}\end{aligned}$$

There is a bijection

$$\{\phi(a, B) : a \in X\} \longrightarrow \{\mathrm{tp}_\phi(a/B) : a \in X\} = S_\phi(B)$$

given by

$$\phi(a, B) \longmapsto \mathrm{tp}_\phi(a/B).$$

It follows that

$$|\mathcal{S}_{\phi^*} \cap B| = |S_\phi(B)|.$$

# Duality in a Model-Theoretic Context

## Lemma

*The dual shatter function of $\phi$ is the shatter function of $\phi^*$.*
*That is $\pi^*_\phi = \pi_{\phi^*}$.*

Proof: For all $n < \omega$, we have

$$\begin{aligned}
\pi^*_\phi(n) &= \max\{|S(\phi(X,b) : b \in B)| : B \in [Y]^n\} \\
&= \max\{|S_\phi(B)| : B \in [Y]^n\} \\
&= \max\{|\mathcal{S}_{\phi^*} \cap B| : B \in [Y]^n\} \\
&= \pi_{\phi^*}(n).
\end{aligned}$$

## Corollary

$\mathrm{VC}^*(\phi) = \mathrm{VC}(\phi^*)$ *and* $\mathrm{vc}^*(\phi) = \mathrm{vc}(\phi^*)$.

# $\mathsf{VC}(\phi) < \omega \iff \mathsf{VC}^*(\phi) < \omega$

## Lemma

$\mathsf{VC}(\phi) < 2^{\mathsf{VC}^*(\phi)+1}$.

Proof: Suppose $\mathsf{VC}(\phi) \geq 2^n$, there exists $A \in [X]^{2^n}$ shattered by $\mathcal{S}_\phi$.

Let $\{a_J : J \subseteq n\}$ enumerate $A$.

For all $i < n$, let $b_i \in Y$ such that $\mathcal{M} \models \phi(a_J, b_i) \iff i \in J$.

Let $B = \{b_i : i < n\}$.

It follows that $\mathcal{S}_{\phi^*}$ shatters $B$, so $\mathsf{VC}(\phi^*) \geq n$.

## Corollary

$\mathsf{VC}^*(\phi) < 2^{\mathsf{VC}(\phi)+1}$.

## Corollary

$\mathsf{VC}(\phi) < \omega \iff \mathsf{VC}^*(\phi) < \omega$.

# Duality in the Classical Context

Given $(X, \mathcal{S})$ a set system, let $\mathcal{M} = (X, \mathcal{S}, \in)$, and $\phi(x, y)$ be $x \in y$.

It follows that $\mathcal{S} = \mathcal{S}_\phi$, so by definition, $\pi_\mathcal{S} = \pi_\phi$ and $\pi^*_\mathcal{S} = \pi^*_\phi$.

Let $X^* = \mathcal{S}$ and

$$\begin{aligned} \mathcal{S}^* &= \{\{B \in \mathcal{S} : a \in B\} : a \in X\} \\ &= \{\phi^*(\mathcal{S}, a) : a \in X\}. \end{aligned}$$

It follows that $\mathcal{S}^* = \mathcal{S}_{\phi^*}$, so by definition, $\pi_{\mathcal{S}^*} = \pi_{\phi^*}$ and $\pi^*_{\mathcal{S}^*} = \pi^*_{\phi^*}$.

**Definition**

We call $(X^*, \mathcal{S}^*)$ the *dual* of $(X, \mathcal{S})$.

**Lemma**

*$\pi^*_\mathcal{S} = \pi_{\mathcal{S}^*}$ and $\pi^*_{\mathcal{S}^*} = \pi_\mathcal{S}$.*

Proof: $\pi^*_\mathcal{S} = \pi^*_\phi = \pi_{\phi^*} = \pi_{\mathcal{S}^*}$ and $\pi^*_{\mathcal{S}^*} = \pi^*_{\phi^*} = \pi_\phi = \pi_\mathcal{S}$.

# Duality in the Classical Context

## Corollary

$\mathsf{VC}^*(\mathcal{S}) = \mathsf{VC}(\mathcal{S}^*)$ *and* $\mathsf{vc}^*(\mathcal{S}) = \mathsf{vc}(\mathcal{S}^*)$.

## Corollary

*For any set system* $(X, \mathcal{S})$, *we have*

$$\mathsf{VC}(\mathcal{S}) < 2^{\mathsf{VC}^*(\mathcal{S})+1}$$

*and*

$$\mathsf{VC}^*(\mathcal{S}) < 2^{\mathsf{VC}(\mathcal{S})+1}.$$

## Corollary

$\mathsf{VC}(\mathcal{S}) < \omega \quad \iff \quad \mathsf{VC}^*(\mathcal{S}) < \omega.$

# References

A. Aschenbrenner, A. Dolich, D. Haskell, H. D. Macpherson, and S. Starchenko, *Vapnik-Chervonenkis density in some theories without the independence property, I*, Trans. Amer. Math. Soc. 368 (2016), 5889-5949.

V. Guingona, *On VC-density in VC-minimal theories*, arXiv:1409.8060 [math.LO].

P. Simon, *A Guide to NIP Theories*, Cambridge University Press (2015).

# Recap: Set Systems in a Model Theoretic Context

Let $\mathcal{L}$ be a language, $\mathcal{M}$ an $\mathcal{L}$-structure, and $\phi(x, y) \in \mathcal{L}$.

$$\mathcal{S}_\phi = \left\{\phi\left(M^{|x|}, b\right) : b \in M^{|y|}\right\}$$

$$\begin{aligned}\pi_\phi(n) &= \max\left\{|\mathcal{S}_\phi \cap A| : A \in \left[M^{|x|}\right]^n\right\} \\ &= \max\left\{|S_{\phi^*}(A)| : A \in \left[M^{|x|}\right]^n\right\}\end{aligned}$$

$$\mathrm{VC}(\phi) = \sup\left\{n < \omega : \pi_\phi(n) = 2^n\right\}$$

$$\mathrm{vc}(\phi) = \inf\left\{r \in \mathbb{R}^{>0} : \pi_\phi(n) = O(n^r)\right\}$$

# Recap: Duality in a Model Theoretic Context

$$S(A_1, \cdots, A_n) = \left\{ \bigcap_{i=1}^{n} A_i^{\sigma(i)} : \sigma \in {}^{n}2 \right\} \setminus \varnothing$$

$$\begin{aligned} \pi^*_\phi(n) &= \max \{ |S(A_1, ..., A_n)| : A_1, ..., A_n \in \mathcal{S}_\phi \} \\ &= \max \left\{ |S_\phi(B)| : B \in \left[ M^{|y|} \right]^n \right\} \end{aligned}$$

$$\mathrm{IN}(\phi) = \mathrm{VC}^*(\phi) = \sup \left\{ n < \omega : \pi^*_\phi(n) = 2^n \right\}$$

$$\mathrm{vc}^*(\phi) = \inf \left\{ r \in \mathbb{R}^{>0} : \pi^*_\phi(n) = O(n^r) \right\}$$

# Elementary Properties

## Lemma

*$\pi^*_\phi$ is elementary (i.e., elementary equivalent $\mathcal{L}$-structures agree on $\pi^*_\phi$).*

Proof: Given $n < \omega$, let $\sigma \in {}^{\mathcal{P}(n)}2$. Consider the $\mathcal{L}$-sentence

$$\exists y_1, \ldots, y_n \bigwedge_{J \subseteq n} \left[ \exists x \bigwedge_{i=1}^{n} \phi^{[i \in J]}(x, y_i) \right]^{\sigma(J)}.$$

## Corollary

$\mathrm{VC}^*(\phi)$ *and* $\mathrm{vc}^*(\phi)$ *are elementary.*

## Corollary

$\mathrm{VC}(\phi)$ *and* $\mathrm{vc}(\phi)$ *are elementary.*

# NIP Formulae

Let $T$ be a complete $\mathcal{L}$-theory, and let $\phi(x, y) \in \mathcal{L}$.

## Definition

We say $\phi$ has the *independence property (IP)* iff: for some $\mathcal{M} \models T$, there exists sequences $(a_J : J \subseteq \omega) \subseteq M^{|x|}$ and $(b_i : i < \omega) \subseteq M^{|y|}$ such that

$$\mathcal{M} \models \phi(a_J, b_i) \iff i \in J.$$

If $\phi$ is not IP, we say $\phi$ is *NIP*.

## Lemma

$\phi$ *is IP* $\iff \mathrm{IN}(\phi) = \omega.$

Proof: Compactness.

## Corollary

$\phi$ *is NIP* $\iff \mathrm{IN}(\phi) < \omega \iff \mathrm{VC}(\phi) < \omega.$

# NIP and $\mathrm{vc}^T$

Let $T$ be a complete $\mathcal{L}$-theory.

**Definition**

We say $T$ is *NIP* iff: every partitioned $\mathcal{L}$-formula is NIP.

Fact: It is sufficient to check all $\phi(x, y)$ with $|x| = 1$.

**Definition**

The *VC density of* $T$ is the function

$$\mathrm{vc}^T : \omega \longrightarrow \mathbb{R}^{\geq 0} \cup \{\infty\}$$

defined by

$$\begin{aligned} \mathrm{vc}^T(n) &= \sup\{\mathrm{vc}(\phi) : \phi(x, y) \in \mathcal{L},\ |y| = n\} \\ &= \sup\{\mathrm{vc}^*(\phi) : \phi(x, y) \in \mathcal{L}, |x| = n\}. \end{aligned}$$

# NIP and $\mathrm{vc}^T$

**Lemma**

*If $\mathrm{vc}^T(n) < \infty$ for all $n < \omega$, then $T$ is NIP.*

Note: Converse is not true in general; e.g., consider $T^{\mathrm{eq}}$ where $T$ is NIP.

Open Questions:

1. For every language $\mathcal{L}$ and every complete $\mathcal{L}$-theory $T$, does $\mathrm{vc}^T(1) < \infty$ imply $\mathrm{vc}^T(n) < \infty$ for all $n < \omega$?
2. If so, is there some bounding function $\beta$, independent of $\mathcal{L}$ and $T$, such that $\mathrm{vc}^T(n) < \beta(\mathrm{vc}^T(1), n)$?

# Finite Types

Let $\Delta(x, y)$ be a finite set of $\mathcal{L}$-formulae (with free variables $x$ and $y$).

## Definition

The set system generated by $\Delta$ is

$$\mathcal{S}_\Delta = \left\{\phi\left(M^{|x|}, b\right) : \phi(x, y) \in \Delta,\ b \in M^{|y|}\right\}.$$

The dual shatter function of $\Delta$ is

$$\pi^*_\Delta(n) = \max\left\{\left|S_\Delta(B)\right| : B \in \left[M^{|y|}\right]^n\right\}.$$

The dual VC density of $\Delta$ is

$$vc^*_\Delta(n) = \inf\{r \in \mathbb{R}^{>0} : \pi^*_\Delta(n) = O(n^r)\}.$$

Fact: $\pi^*_\Delta$ and $vc^*_\Delta$ are elementary.

# Defining Schemata

Let $\Delta(x, y) \subseteq \mathcal{L}$ and $B \subseteq M^{|y|}$ both be finite. Let $p \in S_\Delta(B)$.

## Definition

Given a *schema*

$$d(y, z) = \{d_\phi(y, z) : \phi \in \Delta\} \subseteq \mathcal{L}$$

and a parameter $c \in M^{|z|}$, we say that $d(y, c)$ *defines* $p$ iff:
for every $\phi \in \Delta$ and $b \in B$, we have

$$\phi(x, b) \in p \quad \Longleftrightarrow \quad \mathcal{M} \models d_\phi(b, c).$$

# UDTFS

Let $\Delta(x, y) \subseteq \mathcal{L}$ be finite.

**Definition**

We say $\Delta$ has *uniform definability of types over finite sets (UDTFS) with n parameters* iff: there is a finite family $\mathcal{F}$ of schemata each of the form

$$d(y, z_1, ..., z_n) = \{d_\phi(y, z_1, ..., z_n) : \phi \in \Delta\}$$

with $|y| = |z_1| = \cdots = |z_n|$ such that if $B \subseteq M^{|y|}$ is finite and $p(x) \in S_\Delta(B)$, then for some $d \in \mathcal{F}$ and $b_1, ..., b_n \in B$, $d(y, \overline{b})$ defines $p$.

Fact: This property is elementary.

**Definition**

If $T$ is an $\mathcal{L}$-theory, we say $\Delta$ has *UDTFS in T with n parameters* iff: $\Delta$ has UDTFS with $n$ parameters for all models of $T$.

# Finite Breadth $\Longrightarrow$ UDTFS

Let $\Delta(x, y) \subseteq \mathcal{L}$ be finite.

**Lemma (5.2)**

*If* $\operatorname{breadth}(\mathcal{S}_\Delta) = n < \omega$*, then* $\Delta$ *has UDTFS with* $n$ *parameters.*

Proof: For each $\phi \in \Delta$, let $d^0_\phi(y, z_1, \dots, z_n)$ be $y \neq y$.

For each $\phi \in \Delta$ and each $\overline{\delta} \in \Delta^n$, let $d^{\overline{\delta}}_\phi(y, z_1, \dots, z_n)$ be

$$\forall x \left[ \bigwedge_{i=1}^{n} \delta_i(x, z_i) \longrightarrow \phi(x, y) \right].$$

We claim that the family $\left\{ d^0,\ d^{\overline{\delta}} : \overline{\delta} \in \Delta^n \right\}$ uniformly defines $\Delta$-types over finite sets.

# Proof of Claim:

Let $B \subseteq M^{|y|}$ be finite, and let $p(x) \in S_\Delta(B)$.

If $\forall \phi \in \Delta \quad \forall b \in B \quad \phi(x,b) \notin p$ : $\quad d^0$ defines $p$.

Otherwise:

Let $p\lfloor_\Delta (x) = \{\phi(x,b) \in p : \phi \in \Delta\}$. Since breadth$(\mathcal{S}_\Delta) = n$, there are $\delta_1(x,c_1), ..., \delta_n(x,c_n) \in p\lfloor_\Delta$ such that

$$p(M) \;\subseteq\; p\lfloor_\Delta (M) \;=\; \bigcap_{\phi(x,b)\,\in\, p\lfloor_\Delta} \phi(M,b) \;=\; \bigcap_{i=1}^{n} \delta_i(M,c_i).$$

For all $\phi \in \Delta$ and $b \in B$, we have

$$\phi(x,b) \in p \iff \bigcap \delta_i(M,c_i) \subseteq \phi(M,b) \iff \mathcal{M} \models d_\phi^{\overline{\delta}}(b,\overline{c}).$$

So $d^{\overline{\delta}}(y,\overline{c})$ defines $p$.

# The VC $n$ Property

## Definition

An $\mathcal{L}$-structure has the *VC n property* iff:
all finite $\Delta(x, y) \subseteq \mathcal{L}$ with $|x| = 1$ have UDTFS with $n$ parameters.

Fact: VC $n$ is an elementary property.

## Definition

An $\mathcal{L}$-theory has the *VC n property* iff:
all of its models have VC $n$.

Next goal...

## Theorem (6.1)

*If $T$ is complete and weakly o-minimal, then $T$ has the VC 1 property.*

Let $T$ be an $\mathcal{L}$-theory, and let $\Delta(x,y), \Psi(x,y) \subseteq \mathcal{L}$ both finite.

## Lemma (5.5)

*If every formula in $\Delta$ is $T$-equivalent to a boolean combination of formulae from $\Psi$ and $\Psi$ has UDTFS in $T$ with $n$ parameters, then $\Delta$ has UDTFS in $T$ with $n$ parameters.*

Proof: Let $t = |\Psi|$ and $s = 2^t$. Let $(\psi_j : j < t)$ enumerate $\Psi$.
For each $\phi \in \Delta$, there exists $\sigma \in {}^{s \times t}2$ such that

$$T \vdash \phi(x,y) \longleftrightarrow \bigvee_{i<s} \bigwedge_{j<t} \psi_j^{\sigma(i,j)}(x,y).$$

Let $\mathcal{F}$ witness that $\Psi$ has UDTFS with $n$ parameters.
For each $d \in \mathcal{F}$ and $\phi \in \Delta$, let $d_\phi$ be

$$\bigvee_{i<s} \bigwedge_{j<t} d_{\psi_j}^{\sigma(i,j)}(y, z_1, \ldots z_n).$$

It follows that $\{\{d_\phi : \phi \in \Delta\} : d \in \mathcal{F}\}$ witnesses that $\Delta$ has UDTFS with $n$ parameters.

# Weakly O-Minimal Theories are VC 1

**Theorem (6.1)**

*If $T$ is complete and weakly o-minimal, then $T$ has the VC 1 property.*

Proof: Let $\mathcal{M} \models T$, and let $\Delta(x, y) \subseteq \mathcal{L}$ be finite with $|x| = 1$.

By Compactness, there exists $n < \omega$ such that for all $\phi \in \Delta$ and $b \in M^{|y|}$,

$$\phi(M, b) \text{ has at most } n \text{ maximal convex components.}$$

For all $\phi \in \Delta$ and $i < n$, there exists $\phi_i(x, y) \in \mathcal{L}$ such that for each $b \in M^{|y|}$,

$$\phi_i(M, b) \text{ is the } i^{th} \text{component of } \phi(M, b).$$

It follows that

$$\mathcal{M} \models \phi(x, y) \;\leftrightarrow\; \bigvee_{i<n} \phi_i(x, y).$$

# Proof of Theorem (cont.)

For each $\phi \in \Delta$ and $i < n$, let

$$\begin{array}{lll} \phi_i^{\leq}(x,y) & \text{be} & \exists x_0\,[\phi_i(x_0,y) \ \wedge \ x \leq x_0] \\ \phi_i^{<}(x,y) & \text{be} & \forall x_0\,[\phi_i(x_0,y) \ \rightarrow \ x < x_0]. \end{array}$$

It follows that

$$\mathcal{M} \models \phi_i(x,y) \quad \leftrightarrow \quad \phi_i^{\leq}(x,y) \ \wedge \ \neg\phi_i^{<}(x,y).$$

If we let $\Psi = \{\phi_i^{<},\ \phi_i^{\leq}\ :\ \phi \in \Delta,\ i < n\}$, each formula in $\Delta$ is $T$-equivalent to a boolean combination of $2n$ formulae in $\Psi$.

For each $\psi \in \Psi$ and $b \in M^{|y|}$, notice that $\psi(M,b)$ is an initial segment of $M$, so $\mathcal{S}_\Psi$ is directed.

Lemma 5.2 $\Rightarrow$ $\Psi$ has UDTFS with one parameter.

Lemma 5.5 $\Rightarrow$ $\Delta$ has UDTFS with one parameter.

# Uniform Bounds on VC Density

**Theorem (5.7)**

*If $\mathcal{M}$ has the VC n property, then every finite $\Delta(x, y) \subseteq \mathcal{L}$ has UDTFS with $n|x|$ parameters.*

**Corollary (5.8a)**

*If $\mathcal{M}$ has the VC n property, then for every finite $\Delta(x, y) \subseteq \mathcal{L}$, we have* $\mathrm{vc}^*(\Delta) \leq n|x|$.

Proof: Given $\Delta(x, y)$ finite, there exists finite $\mathcal{F}$ witnessing UDTFS with $n|x|$ parameters. It follows that $|S_\Delta(B)| \leq |\mathcal{F}||B|^{n|x|}$.

**Corollary (5.8b)**

*If $T$ is complete and has the VC n property, then for all $m < \omega$, we have* $\mathrm{vc}^T(m) \leq nm$.

# Uniform Bounds on VC Density

Recall...

## Theorem (6.1)

*If $T$ is complete and weakly o-minimal, then $T$ has the VC 1 property.*

It follows that...

## Corollary (6.1a)

*If $T$ is complete and weakly o-minimal and $\Delta(x, y) \subseteq \mathcal{L}$ is finite, then* $\mathrm{vc}^*(\Delta) \leq |x|$.

## Corollary (6.1b)

*If $T$ is complete and weakly o-minimal, then $\mathrm{vc}^T(n) \leq n$ for all $n < \omega$.*

# Application: RCVF

Let $\mathcal{L} = \{+\,,-\,,\cdot\,,0\,,1\,,<\,,|\}$.

RCVF (with a proper convex valuation ring) where $|$ is the divisibility predicate (i.e., $a|b \;\Leftrightarrow\; v(a) \leq v(b)$) is a complete $\mathcal{L}$-theory.

Cherlin and Dickmann showed RCVF has quantifier elimination and is, therefore, weakly $o$-minimal.

**Corollary (6.2a)**

*In* RCVF, *if* $\Delta(x, y) \subseteq \mathcal{L}$ *is finite, then* $\mathrm{vc}^*(\Delta) \leq |x|$.

**Corollary (6.2b)**

$\mathrm{vc}^{\mathrm{RCVF}}(n) \leq n$ *for all* $n < \omega$.

# Application: $\mathrm{ACVF}_{(0,0)}$

Let $\mathcal{L} = \{+\,,-\,,\cdot\,,0\,,1\,,|\}$.

$\mathrm{ACVF}_{(0,0)}$ where $|$ is the divisibility predicate is complete in $\mathcal{L}$.

Let $R \models \mathrm{RCVF}$ (in $\mathcal{L} \cup \{<\}$).

Consider $R(i)$ where $i^2 = -1$ and

$$a + bi \mid c + di \;\Leftrightarrow\; a^2 + b^2 \mid c^2 + d^2.$$

It follows that $R(i) \models \mathrm{ACVF}_{(0,0)}$ and is interpretable in $R$.

## Corollary (6.3a)

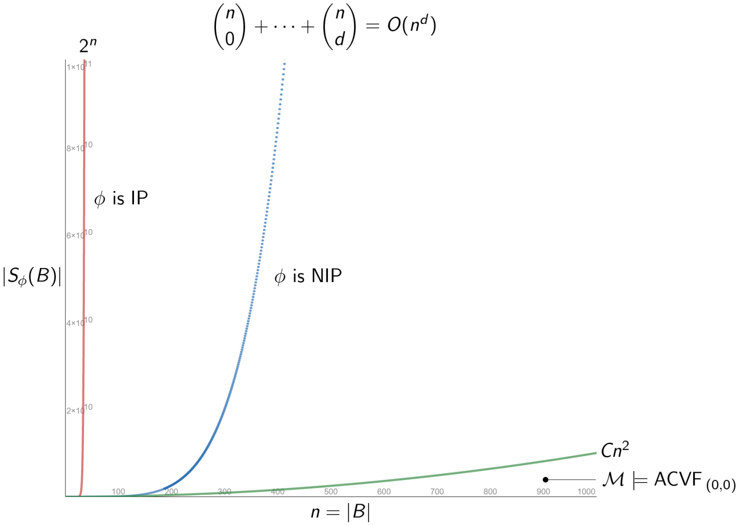
*In* $\mathrm{ACVF}_{(0,0)}$*, if* $\Delta(x,y) \subseteq \mathcal{L}$ *is finite, then* $\mathrm{vc}^*(\Delta) \leq 2|x|$.

## Corollary (6.3b)

$\mathrm{vc}^{\mathrm{ACVF}_{(0,0)}}(n) \leq 2n$*, for all* $n < \omega$.

# Counting Types

Let $\mathcal{L}$ be a language, $\mathcal{M}$ an $\mathcal{L}$-structure, $\phi(x,y) \in \mathcal{L}$ with $|x| = 1$, and $B \subseteq M^{|y|}$.



# Open Questions

1. For every language $\mathcal{L}$ and every complete $\mathcal{L}$-theory $T$, does $\mathrm{vc}^T(1) < \infty$ imply $\mathrm{vc}^T(n) < \infty$ for all $n < \omega$?

$$\begin{array}{rcl} \text{RCVF} & : & \text{Yes} \qquad \text{ACVF}_{(0,p)} : \ ? \\ \text{ACVF}_{(0,0)} & : & \text{Yes} \qquad \text{ACVF}_{(p,p)} : \ ? \end{array}$$

2. If so, is there some bounding function $\beta$, independent of $\mathcal{L}$ and $T$, such that $\mathrm{vc}^T(n) < \beta(\mathrm{vc}^T(1), n)$?

$$\begin{array}{rcl} \text{RCVF} & : & \beta(n) = n \qquad \text{ACVF}_{(0,p)} : \ ? \\ \text{ACVF}_{(0,0)} & : & \beta(n) = 2n \qquad \text{ACVF}_{(p,p)} : \ ? \end{array}$$

3. Is it possible for $\mathrm{vc}(\phi)$ to be irrational?

We summarize the implications between the properties of a theory $T$ discussed above in the following diagram:

$$\begin{tikzcd}
\mathrm{VC}\,1 \arrow[r, Rightarrow] \arrow[d, Rightarrow, "!"'] & \mathrm{VC}\,d \text{ for some } d>0 \arrow[drr, Rightarrow, bend left] & & & \\
\mathrm{vc}^T(1)=1 \arrow[r, Rightarrow] & \text{vc-minimal} \arrow[r, Rightarrow] & \text{dp-minimal} \arrow[r, Rightarrow, "!"] & \text{UDTFS} \arrow[r, Rightarrow] & \text{NIP} \\
\text{VC-minimal} \arrow[u, Rightarrow, "!"] & & & &
\end{tikzcd}$$

Here the arrows marked with an exclamation mark are known not to be reversible.

A. Aschenbrenner, A. Dolich, D. Haskell, H. D. Macpherson, and S. Starchenko, *Vapnik-Chervonenkis density in some theories without the independence property, I*, Trans. Amer. Math. Soc. 368 (2016), 5889-5949.